Title.

Tyrone Cutler

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Abstract

Contents

1 Proper Maps

Proper maps make several appearances in our work. Since they can be very powerful when recognised we include a few results as background. A general reference for this section is Bourbaki [1], §1, Chapter 10.

Definition 1 *A continuous map $f : X \to Y$ is said to be* ***proper*** *if it is closed, and if for each $y \in Y$ the inverse image $f^{-1}(y) \subseteq X$ is compact.* $\square$

Lemma 1.1 *If $f : X \to Y$ is proper and $K \subseteq Y$ is compact, then $f^{-1}(K) \subseteq X$ is compact.*

Proof It will suffice to show that if $\{C_i \subseteq f^{-1}(K)\}_{i\in\mathcal{I}}$ is a family of closed subsets with $\bigcap_\mathcal{I} C_i = \emptyset$, then there is finite subfamily whose intersection is already empty. To this end we consider the family $\{f(C_i) \subseteq K\}_\mathcal{I}$. Since f is proper this is a collection of closed subsets of $K \subseteq Y$ whose total intersection is empty. Since K is compact, there must be finitely many of these sets, say, $f(C_1), \dots, f(C_n)$ such that $f(C_1) \cap \cdots \cap f(C_n) = \emptyset$. But this tells us that $f(C_1 \cap \cdots \cap C_n) = \emptyset$, and the only way this can happen is if $C_1 \cap \cdots \cap C_n = \emptyset$, which was what we needed to show. ∎

Proposition 1.2 *A map $f : X \to Y$ is proper if and only if for each space Z, the map $f \times id_Z : X \times Z \to Y \times Z$ is closed.*

Proof Assume that f is proper. Let Z be a space and let $C \subseteq X \times Z$ closed. We must show that $Q = (f \times id_Z)(C) \subseteq Y \times Z$ is closed. i.e. that any given $(y, z) \in Q^c$ has an open neighbourhood contained in Q^c. If $f^{-1}(y)$ is empty, then we are done, for the image $f(X)$ is closed in Y by assumption, and so a suitable neighbourhood for (y, z) can easily be produced.

So assume that $f^{-1}(y)$ is nonempty. It is compact by assumption, and this means that $f^{-1}(y) \times \{z\}$ is compact subset of $X \times Z$ contained in the (open) complement of C. Appealing to the Tube Lemma we can thus find open sets $U \subseteq X$ and $V \subseteq Z$ such that $f^{-1}(y) \times \{z\} \subseteq U \times V \subseteq C^c$. Then

$$((X \setminus U) \times Z) \cup (X \times (Z \setminus V)) \subseteq X \times Z \tag{1.1}$$

is a closed subset containing C. But the image of this set by $f \times id_Z$ is then a set

$$(f(X \setminus U) \times Z) \cup (f(X) \times (Z \setminus V)) \subseteq X \times Z, \tag{1.2}$$

which contains Q and is seen to be closed by inspection. Since $f(X \setminus U) \subseteq Y$ does not contain y, and $Z \setminus V$ does not contain z, the complement of (1.2) is then an open neighbourhood of (y, z) which is disjoint from Q.

For the converse statement, assume that $f \times id_Z$ is closed for each space Z. By taking $Z = *$ we see that f is closed, so we need only show that $f^{-1}(y)$ is compact for each $y \in Y$. To this end we first prove that if $B \subseteq Y$ is any subspace, then the restriction $g = f| : f^{-1}(B) \to B$ also has the property that $g \times id_Z : f^{-1}(B) \times Z \to B \times Z$ is closed for any space Z. Indeed, if $C \subseteq f^{-1}(B) \times Z$ is closed, then there is some closed subset $\widetilde{C} \subseteq X \times Z$ such that $C = \widetilde{C} \cap (f^{-1}(B) \times Z)$. We have

$$(g \times id_Z)(C) = (f \times id_Z)(\widetilde{C}) \cap B \tag{1.3}$$

and since $(f \times id_Z)(\widetilde{C})$ is closed in $Y \times Z$, it follows that $(g \times id_Z)(C)$ is closed in B.

A consequence of this observation is that if $y \in Y$ is any point, then for any space Z, the restriction $f^{-1}(y) \times Z \to \{y\} \times Z$ is closed. But this is a common characterisation of compactness, so it follows from this that $f^{-1}(y)$ is compact. ∎

Proposition 1.3 *Let $f : X \to Y$ be a map from a Hausdorff space X to a locally compact Hausdorff space Y. Then f is proper if and only if each compact subset $K \subseteq Y$ has a compact preimage under f. If f is proper, then f is locally compact.* ∎

Example 1.1

1) An injective map $f : X \to Y$ is proper if and only if it is closed if and only if it is a closed embedding.

2) Any map $f : X \to Y$ from a compact space X to a Hausdorff space Y is proper.

3) $K \to *$ is proper if and only if K is compact.

4) More generally, for any space X, the projection $pr_X : X \times K \to X$ is proper if and only if K is compact. (Actually this is a consequence of Proposition 1.2 and the last example.)

5) If f, g are both proper, then $f \times g$ is proper. More generally the product of any family of proper maps is proper.

2 Local Compactness

Many different definitions of local compactness appear in the literature. The reader is advised to take care to pay attention to the definition that any given author adopts. The following is that which we shall adhere to in these notes.

Definition 2 *A space X is said to be **locally compact at** a point $x \in X$ if each neighbourhood $U \subseteq X$ of x contains a compact neighbourhood of x. The space X is said to be **locally compact** if it is locally compact at each of its points.* □

If X is locally compact, then each of its points has a compact neighbourhood. Although this is often enough to be useful, the implication is not reversible. There are compact spaces which are not locally compact, and locally compact spaces which are not compact, although, of course, each point of a compact space has a (closed) compact neighbourhood. We give examples in 2.1 which demonstrate that these notions are distinct.

Hausdorff spaces, on the other hand, are more flexible when it comes to local compactness, and most of the common definitions tend to be in agreement for these spaces.

Proposition 2.1 *The following statements are equivalent for a Hausdorff space X.*

1) Every point of X has a compact neighbourhood.

2) Every point of X has a closed compact neighbourhood.

3) X is locally compact.

4) If $U \subseteq X$ is a neighbourhood of a point $x \in X$, then x has a neighbourhood $V \subseteq U$ such that $\overline{V}$ is compact and $\overline{V} \subseteq U$.

5) Each point $x \in X$ has a neighbourhood with compact closure.

6) Each point $x \in X$ has a neighbourhood base consisting of subsets with compact closure.

7) Each point $x \in X$ has a neighbourhood base consisting of compact subsets. ∎

Proof Clearly 4) ⇒ 3) ⇒ 2) ⇒ 1) and 4) ⇒ 7) ⇒ 6) ⇒ 5) so we need only prove two implications.

5) ⇒ 4) Let $x \subseteq X$ and $U \subseteq X$ an open neighbourhood of x. Choose an open neighbourhood V of x with $\overline{V}$ compact. Then $\overline{V} \setminus U$ is a closed set not containing X. Since $\overline{V}$ is compact Hausdorff there exist open sets $W_1, W_2 \subseteq X$ with $W_1 \cap W_2 \cap \overline{V} = \emptyset$ such that $x \in W_1$ and $\overline{V} \setminus U \subseteq W_2 \cap \overline{V}$. Thus $W_1 \cap \overline{V} \subseteq \overline{V} \setminus W_2 \subseteq \overline{V} \cap U \subseteq U$. Then $W_1 \cap V$ is an open neighbourhood of x with compact closure $\overline{W_1 \cap V} \subseteq \overline{W_1 \cap \overline{V}} \subseteq U$.

1) ⇒ 5) If $x \in X$ there is a compact K and open U such that $x \in U \subseteq K \subseteq X$. Then $\overline{U} \subseteq K$ is compact and $x \in U \subseteq \overline{U}$. ∎

Corollary 2.2 *A compact Hausdorff space is locally compact.* ∎

Local compactness is not hereditary in general. However we can show that it is inherited by open and closed subspaces, and we can characterise the locally compact subspaces of a given Hausdorff space.

Proposition 2.3

1) *If X is locally compact and $C \subseteq X$ is closed, then C is locally compact in the subspace topology.*

2) *If X is locally compact and $U \subseteq X$ is open, then U is locally compact in the subspace topology.*

3) *If X is Hausdorff and $A \subseteq X$ is locally compact in the subspace topology, then A is locally closed in X*[1].

4) *If X is locally compact Hausdorff, then a subspace $A \subseteq X$ is locally compact in the subspace topology if and only if it is locally closed.*

Proof 1) Let $V \subseteq C$ be an open neighbourhood of a point $x \in X$. Then there is an open $\widetilde{V} \subseteq X$ such that $C \cap \widetilde{V} = V$. By local compactness we can find an open $U \subseteq X$ and a compact $K \subseteq X$ such that $x \in U \subseteq K \subseteq \widetilde{V}$. Then $C \cap U \subseteq C \cap K \subseteq C \cap \widetilde{V} = V$, so we will be done if we can show that $C \cap K$ is compact in C. Let $\{W_i \subseteq C\}_{i \in \mathcal{I}}$ be an open covering of $C \cap K$ in C. Then there are open sets $\widetilde{W_i} \subseteq X$ such that $C \cap \widetilde{W_i} = W_i$ and $\{\widetilde{W_i}\}_{\mathcal{I}}$ covers $C \cap K$ in X. Then $\{K \cap \widetilde{W_i}\}_{\mathcal{I}}$ covers $C \cap K$ in K, and since $C \cap K$ is closed and hence compact in K, we can find finitely many such sets $C \cap \widetilde{W}_1, \ldots, C \cap \widetilde{W}_n$ which cover $C \cap K$ in K. Then $\widetilde{W}_1, \ldots, \widetilde{W}_n$ covers $C \cap K$ in X and $W_1, \ldots, W_n$ covers $C \cap K$ in C. Thus $C \cap K$ is compact in C.
2) If $x \in U$, then it has a neighbourhood $K \subseteq U$ such that K is compact in X. But this implies that K is compact in U.
3) It suffices to show that each point $x \in A$ has an open neighbourhood $U \subseteq X$ such that $U \cap \overline{A} \subseteq A$. Proceed as follows. By local compactness, the point $x \in A$ has an open neighbourhood $U \subseteq X$ such that $\overline{U} \cap A$ is compact in A (cf. Pr. 2.1). Then $\overline{U} \cap A$ is also compact in X, and hence also here closed. It follows that $\overline{U \cap A} \subseteq \overline{U} \cap A$, and this implies that $U \cap \overline{A} \subseteq \overline{U \cap \overline{A}} \subseteq \overline{U \cap A} \subseteq A$.
4) One direction follows from 3), the other from 1). ∎

Arbitrary coproducts of locally compact spaces are locally compact, but some care must be taken with products. The following lemma will be useful in making a complete statement.

Lemma 2.4 *If $f : X \to Y$ is a continuous open surjection and X is locally compact, then Y is locally compact.*

Proof Let $V \subseteq Y$ be a neighbourhood of a point $y \in Y$. Then $f^{-1}(V)$ contains a compact neighbourhood K of any given point $x \in f^{-1}(y)$. The image $f(K) \subseteq Y$ is then a compact set containing y, and since f is open it contains an open neighbourhood of it. ∎

Proposition 2.5 *Let $\{X_i\}_{i \in \mathcal{I}}$ be a family of nonempty spaces indexed by an arbitrary set $\mathcal{I}$.*

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[1] $A \subseteq X$ is **locally closed** if and only if $A = C \cap U$, where $C \subseteq X$ is closed and $U \subseteq X$ is open. It follows that $A \subseteq X$ is locally closed if and only if each point $x \in A$ has an open neighbourhood $U \subseteq X$ in X such that $U \cap A$ is closed in U. We can also show that $A \subseteq X$ is locally closed if and only if A is open in its closure $\overline{A}$.

1) The coproduct $\coprod_{i\in\mathcal{I}} X_i$ is locally compact if and only if each X_i is locally compact.

2) The product $\prod_{i\in\mathcal{I}} X_i$ is locally compact if and only if each X_i is locally compact and all but finitely many are compact.

Proof 1) The forwards implication follows from Pr. 2.3 and the backwards implication is clear.
2) Assume that $\prod_{\mathcal{I}} X_i$ is locally compact. Since the projections are open surjections, we get from Le. 2.4 that each X_i is locally compact. If $U \subseteq \prod_{\mathcal{I}} X_i$ is open, then by assumption there are $V \subseteq K \subseteq U$, with K compact and V a basic open set. Since the projections map V surjectively onto all but finitely many of the X_i, they also map K onto finitely many of the X_i. It follows from this that all but finitely many of the X_i must be compact.

Let us now prove the converse. It is clear that a finite product of locally compact spaces is locally compact. Since an arbitrary product of compact spaces is compact, we will thus be done if we can show that an arbitrary product of compact, locally compact spaces is locally compact. So assume that all the X_i are both compact and locally compact. It will be enough to show that any basic open neighbourhood U of a given point $x \in \prod_{\mathcal{I}} X_i$ contains a compact neighbourhood of x.

Write U as a finite intersection $\bigcap^n pr_i^{-1}(U_i)$, where $U_i \subseteq X_i$ is open and $pr_i : \prod_{\mathcal{I}} X_i \to X_i$ is the projection. Since each X_i is locally compact we can find a compact $K_i \subseteq U_i$ which contains both $pr_i(x)$ and an open neighbourhood of it. Since each X_i is compact, the projections pr_i are proper, and this implies that $pr^{-1}(K_i) \subseteq \prod_{\mathcal{I}} X_i$ is compact. Then $\bigcap^n pr_i^{-1}(K_i)$ is the desired compact neighbourhood of x. ∎

Corollary 2.6 *Finite products and coproducts of locally compact spaces are locally compact.* ∎

Notice in particular that arbitrary products of compact Hausdorff spaces are locally compact.

Finally here is a useful but somewhat unexpected application of theory developed above. We give a direct proof here, but a more satisfying proof may be obtained later once some aspects of the compact-open topology have been studied.

Proposition 2.7 *Let X be locally compact and $q : Y \to Z$ a quotient map. Then $q \times id_X : Y \times X \to Z \times X$ is a quotient map.*

Proof We need to show that if $U \subseteq Z \times X$ is such that $(q \times id_X)^{-1}(U) \subseteq Y \times X$ is open, then U must be open. To this end assume given a point $(z, x) \in U$, and choose a compact neighbouhood $K \subseteq X$ of x such that $\{z\} \times K \subseteq U$.

Then $W = \{y \in Y \mid \{y\} \times K \subseteq V\} = \{y \in Y \mid q(y) \times K \subseteq (q \times id_X)^{-1}(U)\}$ is nonempty, and by application of the Tube Lemma we can check that it is open. Direct inspection shows that W is saturated with respect to q, and from this we get $(z, x) \in q(W) \times K \subseteq U$. Since K contains an open neighbourhood of x we are done. ∎

We end this section by collecting a set of examples and counterexamples which display some of the subtleties discussed previously.

Example 2.1

1) $\mathbb{R}^n$ is locally compact but not compact.

2) Any discrete space is both locally compact and Hausdorff, but is compact if and only if it is finite.

3) $\mathbb{Q}$ is Hausdorff but not locally compact, since any neighbourhood of any point contains a Cauchy sequence with no convergent subsequent (alternatively $\mathbb{Q}$ is not locally closed in $\mathbb{R}$). In fact this argument show that no point of $\mathbb{Q}$ even has a compact neighbourhood. Since $\mathbb{Q} \subseteq \mathbb{R}$ we see that local compactness is not inherited by arbitrary subspaces.

4) $(\mathbb{R} \times (0, \infty)) \cup \{(0,0)\} \subseteq \mathbb{R}^2$ is Hausdorff and locally compact at each point whose second coordinate is positive. However this space fails to be locally compact at the origin, which has no compact neighbourhood.

5) Each point in the the one-point compactification $\mathbb{Q}_\infty$ of $\mathbb{Q}$ has a closed compact neighbourhood, since $\mathbb{Q}_\infty$ itself is compact. However $\mathbb{Q}_\infty$ is not locally compact, since its open subspace $\mathbb{Q}$ is not. Note that $\mathbb{Q}_\infty$ is not Hausdorff, since $\mathbb{Q}$ is not locally compact.

6) The Hilbert space ℓ^2 of square-summable real sequences is not locally compact. (It suffices to show that the origin has no compact neighbourhood K. Assume it does. Then there is an $\epsilon > 0$, such that $B_\epsilon \subseteq K$, where B_ϵ is the closed ϵ-ball around the origin. But this implies that B_ϵ is compact, and this is a contradiction.)

7) $\coprod_{\mathbb{N}} S^1$ is locally compact, $\prod_{\mathbb{N}} S^1$ is compact Hausdorff, while the wedge point in $\bigvee_{\mathbb{N}} S^1$ does not even have a compact neighbourhood.

8) A quotient of a locally compact space need not be locally compact. For instance the quotient space $X = \mathbb{R}^2/(\mathbb{R} \times 0)$ is not locally compact at the coset $[\mathbb{R} \times 0]$.

9) Any smooth manifold is locally compact. More generally, any locally euclidean space is locally compact.

10) A CW complex is locally compact if and only if it is locally finite.

3 The Compact-Open Topology

For spaces X, Y let $Top(X, Y)$ denote the set of continuous maps $X \to Y$, and for subsets $A \subseteq X$ and $B \subseteq Y$ put

$$W(A, B) = \{f \in Top(X, Y) \mid f(A) \subseteq B\} \subseteq Top(X, Y). \tag{3.1}$$

The following is clear.

Lemma 3.1 *Let X, Y be spaces and $\{A, A_i \subseteq X\}_{i \in \mathcal{I}}$, $\{B, B_j \subseteq Y\}_{j \in \mathcal{J}}$ families of subsets. The following equalities hold.*

1) $W(\bigcup_{\mathcal{I}} A_i, B) = \bigcap_{\mathcal{I}} W(A_i, B)$

2) $W(A, \bigcap_{\mathcal{J}} B_j) = \bigcap_{\mathcal{J}} W(A, B_j)$

3) $\bigcap_{\mathcal{I},\mathcal{J}} W(A_i, B_j) \subseteq W(\bigcup_{\mathcal{I}} A_i, \bigcup_{\mathcal{J}} B_j)$ ∎

The sets $W(K, U)$ cover $Top(X, Y)$ as we let $K \subseteq X$ run over all compact subsets and $U \subseteq Y$ run over all open subsets. Thus these sets are suitable for generating a topology, and the lemma shows they have good properties to do so.

Definition 3 *Let X, Y be spaces. The* ***compact-open topology*** *on $Top(X, Y)$ is that generated by the subbasis*

$$\{W(K, U) \subseteq Top(X, Y) \mid K \subseteq X \text{ compact},\ U \subseteq Y \text{ open}\}. \tag{3.2}$$

We denote by $C(X, Y)$ the resulting space. □

Unfortunately there is a baffling array of different topologies on $Top(X, Y)$ that people study, and there is rarely a 'correct' one to adopt. The compact-open topology turns out to one of the more useful topologies when we impose some (local) compactness conditions. Since many of the spaces we shall ultimately be interested in shall be either compact CW complexes or manifolds - and so shall be locally compact - we shall make exclusive use of it.

To motive the following let us put aside the compact-open topology for a moment and assume that we have devised a similar scheme which assigns to each set $Top(X, Y)$ a topology $\tau = \tau_{X,Y}$. If $C_\tau(X, Y)$ denotes the resulting space, then here is a list of some properties which we would consider desirable for these topologies to possess.

- Functoriality: The assignment $(X, Y) \mapsto C_\tau(X, Y)$ should define a functor $Top \times Top \to Top$.
- Continuity of composition: $C_\tau(Y, Z) \times C_\tau(X, Y) \to C_\tau(X, Z)$, $(g, f) \mapsto g \circ f$, should be a continuous map.
- Adjunction: For fixed X, the functor $Y \mapsto C_\tau(X, Y)$ should be right adjoint to $Z \mapsto Z \times X$.
- Compatibility with homotopy: The homotopy type of $C_\tau(X, Y)$ should depend only on the homotopy types of X, Y.

In practice asking for all these properties to hold is too much. The third item in particular is delicate. As we will see, the compact-open topology does satisfy the first and last listed properties, *but in general the second and third properties fail.* We will, however, be able to identify suitable conditions under which we can recover some useful statements.

We begin by studying functorality and composition. The first of these is immediate, but the second requires a little more work.

Lemma 3.2 *Let $f : X \to Y$ and $g : Y \to Z$ be maps between spaces X, Y, Z. Then the induced functions*

$$g_* : C(X, Y) \to C(X, Z), \qquad h \mapsto h \circ f \tag{3.3}$$

and

$$f^* : C(Y, Z) \to C(X, Z), \qquad h \mapsto h \circ f \tag{3.4}$$

are continuous.

Proof Let $K \subseteq X$ be compact and $V \subseteq Z$ open. Then

$$(g_*)^{-1}(W(K,V)) = W(K, g^{-1}(U)) \tag{3.5}$$
$$(f^*)^{-1}(W(K,V)) = W(f(K), V). \quad \blacksquare \tag{3.6}$$

Addendum Keeping the notation and assumptions of Proposition 3.6 we can show:

1) If g is an embedding, then so is g_*.

2) If f is surjective, then f^* is injective. If moreover f is proper, then f^* is an embedding. $\blacksquare$

Clearly $id_{Y*} = id_{C(X,Y)}$ and we have equalities

$$(h \circ g)_* = h_* \circ g_* \qquad (f \circ k)^* = k^* \circ f^* \tag{3.7}$$

whenever they make sense. This gives us

Corollary 3.3 *The assignment*

$$Top \times Top \xrightarrow{C(-,-)} Top, \qquad (X,Y) \mapsto C(X,Y). \tag{3.8}$$

defines a functor. $\blacksquare$

Unfortunately this does not lead to a topological enrichment of our category. For example, the operation of composition may fail to be continuous.

Proposition 3.4 *Let X, Y, Z be spaces. If Y is locally compact, then the composition map*

$$\circ : C(Y,Z) \times C(X,Y) \to C(X,Z), \qquad (g,f) \mapsto g \circ f, \tag{3.9}$$

is continuous.

Proof We show that $\circ$ is continuous at any point $(g,f) \in C(Y,Z) \times C(X,Y)$. To begin choose a compact $K \subseteq X$ and an open $V \subseteq Z$ such that $g \circ f \in W(K,V) \subseteq C(X,Z)$. Using the continuity of g and the local compactness of Y, for each $y \in f(K)$ we can find neighbourhoods $U_y, L_y \subseteq Y$ such that U_y is open, L_y is compact, $U_y \subseteq L_y$, and $g(L_y) \subseteq V$. Since the U_y cover the compact set $f(K)$, we can find finitely many $U_1, \dots, U_n$ such that $f(K) \subseteq U = \bigcup_{i=1}^n U_i$. If $L_1, \dots, L_n$ are the corresponding compact sets, then $L = \bigcup_{i=1}^n L_i$ is compact, $U \subseteq L$, and $g(L) \subseteq V$.

Now $(g,f) \in W(L,V) \times W(K,U)$, and if $(g', f') \in W(L,V) \times W(K,U)$ is another pair, then

$$g'(f'(K)) \subseteq g'(U) \subseteq g'(L) \subseteq V \tag{3.10}$$

which shows that

$$\circ\,(W(L,V) \times W(K,U)) \subseteq W(K,V) \tag{3.11}$$

completing the proof. $\blacksquare$

Moving onwards, let us now consider conditions under which the functors $(-) \times Y$ and $C(Y, -)$ form an adjoint pair. Notice that if the functor $(-) \times Y$ has a right adjoint at all, then it must necessarily send a space Z to the set $Top(Y, Z)$ endowed with *some* topology. This can be seen by evaluating $(-) \times Y$ at the one-point space.

In general $(-) \times Y$ will not have a right adjoint. For instance, when $Y = \mathbb{Q}$ it does not [3]. Although an answer to the full existence question is beyond the scope of these notes, we will be able to locate a good class of spaces Y for which this right adjoint both exists, and coincides with $C(Y, -)$ in the compact-open topology. The class we have in mind will turn out to be that of the locally compact spaces in the sense of Def. 2.

So, reformulating our question it becomes: for which Y is continuity of a map $X \times Y \to Z$ equivalent to continuity of its adjoint $X \to C(Y, Z)$ for any given spaces X, Z? Our partial answer to this will make use of the following tool. For spaces X, Y define a set-valued function $ev = ev_{X,Y}$ by setting

$$ev : C(X, Y) \times X \to Y, \qquad (f, x) \mapsto f(x). \tag{3.12}$$

In general this map is not continuous. We call ev the **evaluation map**, and next study conditions which guarantee its continuity.

Lemma 3.5 *If X, Y are spaces with X nonempty, then the map*

$$c_{-} : Y \to C(X, Y), \qquad y \mapsto [c_y : x \mapsto y] \tag{3.13}$$

is an embedding.

Proof Clearly c is an injection of sets. It is also continuous, since if $K \subseteq X$ is compact and $U \subseteq Y$ is open, then $c^{-1}(W(K, U)) = U$. Now, if $U \subseteq Y$ is open and $x \in X$, then

$$c(U) = W(\{x\}, U) \cap c(Y) \tag{3.14}$$

and this shows that c is an embedding, since $W(\{x\}, U) \subseteq C(X, Y)$ is open. ∎

Addendum If X is nonempty and Y is Hausdorff, then $c : Y \to C(X, Y)$ is a closed embedding. ∎

Corollary 3.6 *If X is locally compact and Y is any space, then the evaluation map*

$$ev : C(X, Y) \times X \to Y, \qquad (f, x) \mapsto f(x) \tag{3.15}$$

is continuous.

Proof The composition map

$$\circ : C(X, Y) \times C(*, X) \to C(*, Y) \tag{3.16}$$

is continuous according to Proposition 3.4. According to Lemma (3.5), there are homeomorphisms

$$c_X : X \xrightarrow{\cong} C(*, X), \qquad c_Y : Y \xrightarrow{\cong} C(*, Y) \tag{3.17}$$

and under these identifications, the map (3.16) is exactly the evaluation (3.15). ∎

Given a map $f : X \times Y \to Z$ we define its **right adjoint** to be the function of sets

$$f^{\#} : X \to C(Y,Z), \qquad x \mapsto [f^{\#}(x) : y \to f(x,y)]. \tag{3.18}$$

Similarly, given a map $g : X \to C(Y,Z)$ we define its **left adjoint** to be the function of sets

$$g^{\flat} : X \times Y \to Z, \qquad (x,y) \mapsto g(x)(y). \tag{3.19}$$

In general it will be clear which of left or right adjoint we mean, and we will just say **adjoint** to mean either. We stress that the adjoints need not be continuous. However, they do behave well with respect to composition.

Lemma 3.7 *Let $f : X \times Y \to Z$ and $g : X \to C(Y,Z)$ be given. If $h : X' \to X$, $k : Y' \to Y$ and $l : Z \to Z'$ are maps, then*

$$(lf(h \times k))^{\#} = l_* h^* f^{\#} h : X' \to C(Y', Z') \tag{3.20}$$

$$(l_* k^* g h)^{\flat} = l g^{\flat}(h \times k) : X' \times Y' \to Z'. \tag{3.21}$$

In particular, if h, k, l are continuous, then $(lf(h \times k))^{\#}$ is continuous if $f^{\#}$ is continuous, and $(l_* g h)^{\flat}$ is continuous if $g^{\flat}$ is continuous. ∎

More generally we have the following useful statements.

Proposition 3.8 *Let X, Y, Z be spaces. If $f : X \times Y \to Z$ is continuous, then the right adjoint*

$$f^{\#} : X \to C(Y,Z) \tag{3.22}$$

is also continuous. If Y is locally compact and $g : X \to C(Y,Z)$ is continuous, then the left adjoint

$$g^{\flat} : X \times Y \to Z \tag{3.23}$$

is also continuous.

Proof We show that $f^{\#}$ is continuous at any given point $x \in X$. For this it is sufficient to show that if $K \subseteq Y$ is compact and $V \subseteq Y$ open such that $f^{\#}(x) \in W(K,V) \subseteq C(Y,Z)$, then there is an open neighbourhood $U \subseteq X$ of x such that $f^{\#}(U) \subseteq W(K,V)$.

Now, $f^{-1}(V) \subseteq X \times Y$ is open, and $\{x\} \times K \subseteq f^{-1}(V)$. Let $U \subseteq X$ be any open set satisfying $\{x\} \times K \subseteq U \times K \subseteq f^{-1}(U)$. Note that such a set always exists according to the Tube Lemma. Then for each $x' \in U$ we have $f^{\#}(x')(K) \subseteq V$, which implies that $f^{\#}(U) \subseteq W(K,V)$, which was exactly what we needed to show.

For the second statement we write $g^{\flat}$ as the composition

$$g^{\flat} : X \times Y \xrightarrow{g \times 1} C(Y,Z) \times Y \xrightarrow{ev} Z. \tag{3.24}$$

If Y is locally compact, then this is continuous by Proposition 3.6. ∎

For any spaces X, Y, Z be spaces we get from Proposition 3.8an injection of sets

$$(-)^{\#} : Top(X \times Y, Z) \to Top(X, C(Y, Z)), \qquad f \mapsto f^{\#}. \tag{3.25}$$

Here injectivity follows, since if $f, g : X \times Y \to Z$ have $f^{\#} = g^{\#}$, then $f(x, y) = g(x, y)$ for all $(x, y) \in X \times Y$, so that $f = g$. In general this map fails to be bijective, since a given map $h : X \to C(Y, Z)$ may have a discontinuous adjoint $h^{\flat} : X \times Y \to Z$. On the other hand, the proof of 3.8 shows that $h^{\flat}$ will be continuous whenever $ev : C(Y, Z) \times Y \to Z$ is. Combining this with Le. 3.5, we record the following.

Proposition 3.9 *For spaces X, Y, Z the function*

$$(-)^{\#} : Top(X \times Y, Z) \to Top(X, C(Y, Z)), \qquad f \mapsto f^{\#} : x \mapsto [y \mapsto f(x, y)] \tag{3.26}$$

is an injection of sets, which is natural in each variable separately. If the evaluation map $ev_{Y,Z} : C(Y, Z) \times Y \to Z$, is continuous, then $(-)^{\#}$ is a bijection of sets, natural in X and Z, and

$$(-)^{\flat} : Top(X, C(Y, Z)) \to Top(X \times Y, Z), \qquad g \mapsto [g^{\flat} : (x, y) \mapsto g(x)(y)] \tag{3.27}$$

is its set-theoretic inverse. In particular these functions are inverse bijections when Y is locally compact. ∎

It is not necessary for Y to be locally compact for $ev : C(X, Y) \times Y \to Z$ to be continuous. Indeed, Hofmann and Lawson [4] have constructed a non-locally compact space for which it is. The full story involves something we will not define which is called *core compactness* [2] which generalised local compactness.

On the other hand, here is an example to demonstrate that *some* form of compactness conditions are clearly necessary.

Example 3.1 ($C(\mathbb{Q}, I)$) Take $X = \mathbb{Q}$, $Y = I$ and consider the space $C(\mathbb{Q}, I)$ of continuous maps $\mathbb{Q} \to I$ in the compact-open topology. In the subspace topology $\mathbb{Q} \subseteq \mathbb{R}$, the rationals are not locally compact, so in particular Proposition 3.6 does not guarantee that the evaluation map $ev : C(\mathbb{Q}, I) \times \mathbb{Q} \to I$ is continuous. In fact this map is not continuous, and we can prove this by showing that it is not continuous at (c_1, q), where $c_1 : \mathbb{Q} \to I$, $x \mapsto 1$, is the constant map at 1 and $q \in \mathbb{Q}$ is any point.

Indeed, if ev is continuous at (c_1, q), then there must be a basic open neighbourhood $B \subseteq C(\mathbb{Q}, I)$ of c_1, and an open neighbourhood $N \subseteq \mathbb{Q}$ of q such that $ev(B \times N)$ is contained in $(0, 1]$. Assume this is so and choose compact $K_1, \ldots, K_n \subseteq \mathbb{Q}$ and open $U_1, \ldots, U_n \subseteq I$ such that $B = \cap_{i=1,\ldots,n} W(K_i, U_i)$.

Then $K = \bigcup_{i=1,\ldots,n} K_i$ is compact, and in particular closed in $\mathbb{Q}$. This implies that N cannot be contained inside K, for if it were it would have compact closure $\overline{N}$, and this would imply that $\overline{N}$ would be closed in $\mathbb{R}$, which clearly cannot be true.

Thus there is a point $x \in N \setminus K$, and since $\mathbb{Q}$ is completely regular, also a continuous function $f : \mathbb{Q} \to I$ with $f(x) = 0$ and $f(K) = \{1\}$. Clearly $f \in B$. However here we encounter a problem, for this implies that $ev^{-1}((0, 1])$ does not contain $B \times N$ since it does not even contain (f, x). Thus we have arrived at a contradiction, and it follows that $ev : C(\mathbb{Q}, I) \times \mathbb{Q} \to I$ cannot be continuous at (c_1, q). □

3.1 Further Properties of the Compact-Open Topology

In this section we collect some results on the compact-open topology which are frequently useful to know. While we include statements like Le. 3.11 only for completeness, we will assume results like Pr. 3.12 to be known.

Proposition 3.10 *If X, Y are spaces and Y is T_i for $i \in \{0, 1, 2, 3\}$, then $C(X, Y)$ is T_i.*

Proof (T_0, T_1) Let $f \neq g \in C(X, Y)$. Then there exists $x \in X$ such that $f(x) \neq g(x)$. If Y is T_0, then we can find an open subset $U \subseteq Y$ with, say, $f(x) \in U$ and $g(x) \notin U$. Then $f \in W(\{x\}, U)$ and $g \notin W(\{x\}, U)$, which proves that $C(X, Y)$ is T_0. If we assume instead that Y is T_1, then the same argument shows that $C(X, Y)$ is T_1.
(T_2) Assume now that Y is Hausdorff. If $f, g \in C(X, Y)$ and there is $x \in X$ such that $f(x) \neq g(x)$, then we can find disjoint open neighbourhoods U_f of $f(x)$ and U_g of $f(g)$. Then $f \in W(\{x\}, U_f)$ and $g \in W(\{x\}, U_g)$ and $W(\{x\}, U_f) \cap W(\{x\}, U_g) = \emptyset$, which proves that $C(X, Y)$ is Hausdorff.
(T_3) Now assume that Y is T_3. To show that $C(X, Y)$ is regular it will suffice to show for any basic open neighbourhood $W \subseteq C(X, Y)$ of a given map f, there is another basic open subset $W' \subseteq W \subseteq C(X, Y)$ such that $f \in W'$ and $\overline{W}' \subseteq W$.

First consider the case that W is subbasic. That is, that there exists a compact subset $K \subseteq X$ and an open subset $U \subseteq Y$ such that $W = W(K, U)$. Then we have $f(K) \subseteq U$, and since Y is regular we can find an open subset V such that $f(K) \subseteq V$ and $\overline{V} \subseteq U$. In particular this gives us that $f \in W(K, V) \subseteq W(K, \overline{V}) \subseteq W(K, U)$.

Next we show that $\overline{W(K, V)} \subseteq W(K, \overline{V})$. To this end, let $g \in W(K, \overline{V})^c$. This means that there exists a point $x \in K$ such that $g(x) \notin \overline{V}$ and also that $g \in W(\{x\}, Y \setminus \overline{V})$. In particular $W(\{x\}, Y \setminus \overline{V}) \cap W(K, V) = \emptyset$, so $W(\{x\}, Y \setminus \overline{V})$ is an open neighbourhood of g disjoint from $W(K, V)$, showing that $g \notin \overline{W(K, V)}$. It follows from this that $\overline{W(K, V)} \subseteq W(K, \overline{V})$.

Finally we prove the case that W is basic. Let $K_1, \ldots, K_n \subseteq X$ be compact and $U_1, \ldots, U_n \subseteq Y$ open such that $f \in W = \bigcap_{i=1,\ldots,n} W(K_i, U_i)$. For each $i = 1, \ldots, n$ take $W(K_i, U_i)$ and repeat the previous steps to find an open subset $V_i \subseteq U_i$ with $f \in V_i$ and $\overline{V}_i \subseteq U_i$. Then $\overline{W(K_i, V_i)} \subseteq W(K_i, \overline{V}_i)$ for each $i = 1, \ldots, n$, and $f \in \bigcap_{i=1,\ldots,n} W(K_i, V_i) \subseteq \overline{\bigcap_{i=1,\ldots,n} W(K_i, V_i)} \subseteq \bigcap_{i=1,\ldots,n} W(K_i, U_i)$, which is what we needed to show. ∎

The following lemma is handy in particular when X, Y are metric spaces.

Lemma 3.11 *Let X be a Hausdorff space and Y a space. If $\mathcal{U}$ is a subbase for the topology on Y, then the family*

$$\mathcal{W} = \{W(K, U) \mid K \subseteq X \textit{ compact}, U \in \mathcal{U}\} \tag{3.28}$$

is a subbase for the compact-open toplogy on $C(X, Y)$. ∎

Proof It suffices to show that the subbasic open sets in the compact-open topology are open in the topology on $C(X, Y)$ which is generated by the subbasis $\mathcal{W}$. Thus let $V \subseteq Y$ be open, $K \subseteq X$ compact, and consider $W(K, V)$. We will construct a suitable open neighbourhood for a given map $f \in W(K, V)$.

To begin notice that since $\mathcal{U}$ is a subbasis for Y there is a family of open sets $\{W_a \subseteq Y\}_{a\in\mathcal{A}}$ such that $V = \bigcup_{a\in\mathcal{A}} W_a$ and each W_a is an intersection of finitely many sets in $\mathcal{U}$. Since $f(K) \subseteq V$, the sets $f^{-1}(W_a)$ cover K. Since X is Hausdorff, the compact subset K is regular, so if $x \in K \cap f^{-1}(W_a)$, then there is an open set $U_x \subseteq K$ such that $x \in U_x \subseteq \overline{U}_x \subseteq K \cap f^{-1}(W_a)$. Notice that each $\overline{U}_x$ is compact. Now the open sets U_x cover K, so by compactness we can find finitely many $U_1, \dots, U_n$ whose union contains K. Let $W_1, \dots, W_n$ be the corresponding opens of Y, so that $U_i \subseteq f^{-1}(W_i)$ for each i. This gives $f \in W(\overline{U}_i, W_i)$ for each i.

Next, for each $i = 1, \dots, n$ we can finitely many subbasic sets $V_{i1}, \dots, V_{im_i} \in \mathcal{U}$ such that $W_i = \bigcap_{j=1}^{m_i} V_{ij}$. This gives

$$f \in \bigcap_{i=1}^{n}\bigcap_{j=1}^{m_j} W(\overline{U}_i, V_{ij}) = \bigcap_{i=1}^{n} W(\overline{U}_i, W_i) \subseteq W(K, \bigcup_i W_i) \subseteq W(K, V). \tag{3.29}$$

The intersection on the left is open in the topology on $C(X, Y)$ which is generated by $\mathcal{W}$, so this inclusion is exactly what we were looking for. ∎

Proposition 3.12 *The following statements hold.*

1) *If $\{X_i\}_{i\in\mathcal{I}}$ is a collection of spaces and Y is a space, then the canonical map $C(\coprod_{\mathcal{I}} X_i, Y) \xrightarrow{\cong} \prod_{\mathcal{I}} C(X_i, Y)$ is a homeomorphism.*

2) *If X is a space and $\{Y_i\}_{i\in\mathcal{I}}$ a family of spaces, then the canonical map $C(X, \prod_{\mathcal{I}} Y_i) \to \prod C(X, Y_i)$ is a continuous bijection. If X is locally compact, then this map is a homeomorphism.*

Proof 1) The canonical map is induced by the family of inclusions $X_i \hookrightarrow \coprod_{\mathcal{I}} X_i$, so is continuous by Pr. 3.4, and is bijective by the universal property of the coproduct. Since the compact subsets of $\coprod_{\mathcal{I}} X_i$ are disjoint unions of finitely many compact subsets $K_i \subseteq X_i$, we see using Le. 3.1 that $C(\coprod_{\mathcal{I}} X_i, Y)$ has a subbase given by the sets $W(K_i, U)$, where $K_i \subseteq X_i$ is compact and $U_i \subseteq Y$ is open. Clearly these sets get sent to subasic open sets in $\prod_{\mathcal{I}} C(X_i, Y)$. Thus the canonical map is open and bijective.
2) The canonical map is induced by the projections $pr_i : \prod_{\mathcal{I}} X_i \to X_i$, so is continuous by Pr. 3.4, and is bijective by the universal property of the product. In the case that X is locally compact we get a continuous inverse by taking the adjoint of the composition

$$\begin{aligned} X \times \prod_{i\in\mathcal{I}} C(X, Y_i) &\xrightarrow{\Delta\times 1} \left(\prod_{i\in\mathcal{I}} X\right) \times \left(\prod_{i\in\mathcal{I}} C(X, Y_i)\right) \\ &\xrightarrow{\cong} \prod_{i\in\mathcal{I}} (X \times C(X, Y_i)) \\ &\xrightarrow{\prod ev} \prod_{i\in\mathcal{I}} Y_i \end{aligned} \tag{3.30}$$

where Δ is the diagonal. ∎

Remark If X is a discrete space, then Pr. 3.12 gives a homeomorphism $C(X, Y) \cong \prod_{x\in X} Y$. □

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